Programming Languages: Functional Programming
Practicals 7. Types and Logic

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1. Prove the following propositions:

   (a) $P \to Q \to P$.

   Solution:

   $$\dfrac{\dfrac{\dfrac{P \in \{P, Q\}}{P, Q \vdash P}\ \text{Hyp}}{P \vdash Q \to P}\ {\Rightarrow}\text{I}}{\vdash P \to Q \to P}\ {\Rightarrow}\text{I}$$

   (b) $(P \to Q \to R) \to Q \to P \to R$.

   Solution: Abbreviate $P \to Q \to R$, Q, P to Γ.

   $$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P \to Q \to R \in \Gamma}{\Gamma \vdash P \to Q \to R}\ \text{Hyp} \qquad \dfrac{P \in \Gamma}{\Gamma \vdash P}\ \text{Hyp}}{\Gamma \vdash Q \to R}\ {\Rightarrow}\text{E} \qquad \dfrac{Q \in \Gamma}{\Gamma \vdash Q}\ \text{Hyp}}{\Gamma \vdash R}\ {\Rightarrow}\text{E}}{P \to Q \to R, Q \vdash P \to R}\ {\Rightarrow}\text{I}}{P \to Q \to R \vdash Q \to P \to R}\ {\Rightarrow}\text{I}}{\vdash (P \to Q \to R) \to Q \to P \to R}\ {\Rightarrow}\text{I}$$

   (c) $(P \to Q) \to (Q \to R) \to P \to R$.

   Solution: Abbreviating $P \to Q$, $Q \to R$, P to Γ:

   $$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Q \to R \in \Gamma}{\Gamma \vdash Q \to R}\ \text{Hyp} \qquad \dfrac{\dfrac{P \to Q \in \Gamma}{\Gamma \vdash P \to Q}\ \text{Hyp} \qquad \dfrac{P \in \Gamma}{\Gamma \vdash P}\ \text{Hyp}}{\Gamma \vdash Q}\ {\Rightarrow}\text{E}}{P \to Q, Q \to R, P \vdash R}\ {\Rightarrow}\text{E}}{P \to Q, Q \to R \vdash P \to R}\ {\Rightarrow}\text{I}}{P \to Q \vdash (Q \to R) \to P \to R}\ {\Rightarrow}\text{I}}{\vdash (P \to Q) \to (Q \to R) \to P \to R}\ {\Rightarrow}\text{I}$$

(d) $P \to (P \to Q) \to (P \to Q \to R) \to R$.

Solution: Abbreviating $P, P \to Q, P \to Q \to R$ to Γ:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P \to Q \to R \in \Gamma}{\Gamma \vdash P \to Q \to R}\,\text{Hyp} \qquad \dfrac{P \in \Gamma}{\Gamma \vdash P}\,\text{Hyp}}{\Gamma \vdash Q \to R}\,{\Rightarrow}\text{E} \qquad \dfrac{\dfrac{P \to Q \in \Gamma}{\Gamma \vdash P \to Q}\,\text{Hyp} \qquad \dfrac{P \in \Gamma}{\Gamma \vdash P}\,\text{Hyp}}{\Gamma \vdash Q}\,{\Rightarrow}\text{E}}{P,\ P \to Q,\ P \to Q \to R \vdash R}\,{\Rightarrow}\text{E}}{P,\ P \to Q \vdash (P \to Q \to R) \to R}\,{\Rightarrow}\text{I}}{P \vdash (P \to Q) \to (P \to Q \to R) \to R}\,{\Rightarrow}\text{I}}{\vdash P \to (P \to Q) \to (P \to Q \to R) \to R}\,{\Rightarrow}\text{I}$$

(e) $(P \to Q \to R) \to (P \land Q) \to R$.

Solution: Abbreviate $P \to Q \to R,\ P \land Q$ to Γ.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P \to Q \to R \in \Gamma}{\Gamma \vdash P \to Q \to R}\,\text{Hyp} \qquad \dfrac{\dfrac{P \land Q \in \Gamma}{\Gamma \vdash P \land Q}\,\text{Hyp}}{\Gamma \vdash P}\,\land\text{E}}{\Gamma \vdash Q \to R}\,{\Rightarrow}\text{E} \qquad \dfrac{\dfrac{P \land Q \in \Gamma}{\Gamma \vdash P \land Q}\,\text{Hyp}}{\Gamma \vdash Q}\,\land\text{E}}{P \to Q \to R,\ P \land Q \vdash R}\,{\Rightarrow}\text{E}}{P \to Q \to R \vdash (P \land Q) \to R}\,{\Rightarrow}\text{I}}{\vdash (P \to Q \to R) \to (P \land Q) \to R}\,{\Rightarrow}\text{I}$$

(f) $(P \land Q) \to ((P \lor Q) \to R) \to R)$.

Solution: Abbreviate $P \land Q, (P \lor Q) \to R$ to Γ.

$$\dfrac{\dfrac{\dfrac{\dfrac{(P \lor Q) \to R \in \Gamma}{\Gamma \vdash (P \lor Q) \to R}\,\text{Hyp} \qquad \dfrac{\dfrac{\dfrac{P \land Q \in \Gamma}{\Gamma \vdash P \land Q}\,\text{Hyp}}{\Gamma \vdash P}\,\land\text{E}}{\Gamma \vdash P \lor Q}\,\lor\text{I}}{P \land Q, (P \lor Q) \to R \vdash R}\,{\Rightarrow}\text{E}}{P \land Q \vdash ((P \lor Q) \to R) \to R}\,{\Rightarrow}\text{I}}{\vdash (P \land Q) \to ((P \lor Q) \to R) \to R}\,{\Rightarrow}\text{I}$$

Alternatively, instead of $\Gamma \vdash P$, you can also produce $\Gamma \vdash Q$ under the $\land E$ rule on the righthand branch.

(g) $(P \to Q \to R) \to (P \to Q) \to P \to R$.

Solution: Abbreviate $P \to Q \to R, P \to Q, P$ to Γ.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P \to Q \to R \in \Gamma}{\Gamma \vdash P \to Q \to R}\,\mathsf{Hyp} \quad \dfrac{P \in \Gamma}{\Gamma \vdash P}\,\mathsf{Hyp}}{\Gamma \vdash Q \to R}\,{\Rightarrow}\mathsf{E} \quad \dfrac{\dfrac{P \to Q \in \Gamma}{\Gamma \vdash P \to Q}\,\mathsf{Hyp} \quad \dfrac{Q \in \Gamma}{\Gamma \vdash Q}\,\mathsf{Hyp}}{\Gamma \vdash Q}\,{\Rightarrow}\mathsf{E}}{P \to Q \to R, P \to Q, P \vdash R}\,{\Rightarrow}\mathsf{E}}{P \to Q \to R, P \to Q \vdash P \to R}\,{\Rightarrow}\mathsf{I}}{P \to Q \to R \vdash (P \to Q) \to P \to R}\,{\Rightarrow}\mathsf{I}}{\vdash (P \to Q \to R) \to (P \to Q) \to P \to R}\,{\Rightarrow}\mathsf{I}$$

2. Reduce the following expressions to normal form, if possible.

   (a) $(\lambda x\,.\,x + x)\;3$.

   Solution: $(\lambda x\,.\,x + x)\;3 \;\xrightarrow{\beta}\; 3 + 3 \;\xrightarrow{\beta}\; 6.$

   (b) $(\lambda f\,x\,.\,f\;x\;x)\;(\lambda y\,z\,.\,y + z)\;3$.

   Solution:

$$\begin{aligned} & (\lambda f\,x\,.\,f\;x\;x)\;(\lambda y\,z\,.\,y + z)\;3 \\ \xrightarrow{\beta}\; & (\lambda x\,.\,(\lambda y\,z\,.\,y + z)\;x\;x)\;3 \\ \xrightarrow{\beta}\; & (\lambda y\,z\,.\,y + z)\;3\;3 \\ \xrightarrow{\beta}\; & (\lambda z\,.\,3 + z)\;3 \\ \xrightarrow{\beta}\; & 3 + 3 \\ \xrightarrow{\beta}\; & 6\;. \end{aligned}$$

   (c) $(\lambda x\,.\,x\;x)\;(\lambda x\,.\,x)$.

   Solution:

$$\begin{aligned} & (\lambda x\,.\,x\;x)\;(\lambda x\,.\,x) \\ \xrightarrow{\beta}\; & (\lambda x\,.\,x)\;(\lambda x\,.\,x) \\ \xrightarrow{\beta}\; & (\lambda x\,.\,x)\;. \end{aligned}$$

   (d) $(\lambda x\,.\,x\;x)\;(\lambda x\,.\,x\;x)$.

Solution:

$$
\begin{aligned}
& (\lambda x \,.\, x\; x)\; (\lambda x \,.\, x\; x) \\
\xrightarrow{\beta}\; & (\lambda x \,.\, x\; x)\; (\lambda x \,.\, x\; x) \\
\xrightarrow{\beta}\; & \ldots
\end{aligned}
$$

This term keeps reducing to itself and does not reduce to a normal form.

(e) $(\lambda x \,.\, f\; (x\; x))\; (\lambda x \,.\, f\; (x\; x))$.

Solution:

$$
\begin{aligned}
& (\lambda x \,.\, f\; (x\; x))\; (\lambda x \,.\, f\; (x\; x)) \\
\xrightarrow{\beta}\; & f\; ((\lambda x \,.\, f\; (x\; x))\; (\lambda x \,.\, f\; (x\; x))) \\
\xrightarrow{\beta}\; & f\; (f\; ((\lambda x \,.\, f\; (x\; x))\; (\lambda x \,.\, f\; (x\; x)))) \\
\xrightarrow{\beta}\; & f\; (f\; (f\; ((\lambda x \,.\, f\; (x\; x))\; (\lambda x \,.\, f\; (x\; x))))) \\
\xrightarrow{\beta}\; & \ldots
\end{aligned}
$$

This term keeps producing f. It can be used to find the fixed-point of f. (Keyword: “Y combinator”.)

3. Write down the type derivation trees of the following expressions, if possible.

(a) $(\lambda x\, y \,.\, x)$.

Solution:

$$
\cfrac{\cfrac{\cfrac{x :: a \in \{x :: a, y :: b\}}{x :: a, y :: b \vdash x :: a}\;\textsf{Var}}{x :: a \vdash (\lambda y \,.\, x) :: b \to a}\;{\to}\textsf{I}}{\vdash (\lambda x\, y \,.\, x) :: a \to b \to a}\;{\to}\textsf{I}
$$

(b) $(\lambda p \,.\, (\mathit{snd}\; p, \mathit{fst}\; p))$.

Solution: Abbreviate $p :: (a, b)$ to Γ.

$$
\cfrac{\cfrac{\cfrac{\cfrac{p :: (a,b) \in \Gamma}{\Gamma \vdash p :: (a,b)}\;\textsf{Var}}{\Gamma \vdash \mathit{snd}\; p :: b}\;\wedge\textsf{E} \qquad \cfrac{\cfrac{p :: (a,b) \in \Gamma}{\Gamma \vdash p :: (a,b)}\;\textsf{Var}}{\Gamma \vdash \mathit{fst}\; p :: a}\;\wedge\textsf{E}}{p :: (a,b) \vdash (\mathit{snd}\; p, \mathit{fst}\; p)}\;\wedge\textsf{I}}{\vdash (\lambda p \,.\, (\mathit{snd}\; p, \mathit{fst}\; p)) :: (a,b) \to (b,a)}\;{\to}\textsf{I}
$$

(c) $(\lambda f\, g\, x \,.\, f\ x\ (g\ x))$.

Solution: See the handouts.

(d) $(\lambda x \,.\, x\ x)\ (\lambda x \,.\, x\ x)$.

Solution: It is not possible to type this term in simply-typed λ-calculus. If you attempt to give it a type, you will see that the type of x has to be $((\ldots \to a) \to a) \to a$.
Note also that this term does not have a normal form. One important result is that all typable terms in simply-typed λ-calculus has a normal form.

4. Given the following types, construct (simply typed) lambda expressions having the types.

(a) $(P \to Q \to R) \to Q \to P \to R$.

Solution: $(\lambda f\, x\, y \,.\, f\ y\ x)$.

(b) $(P \wedge Q) \to ((P \vee Q) \to R) \to R$.

Solution: $(\lambda p\, f \,.\, f\ (\text{Left}\ (\mathit{fst}\ p)))$. Another possibility is $(\lambda p\, f \,.\, f\ (\text{Right}\ (\mathit{snd}\ p)))$.