# Programming Languages: Imperative Program Construction
# Practicals 1: Non-Looping Constructs and Weakest Precondition

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**Guarded Command Language Basics**

1. Which of the following Hoare triples hold?
   (a) $\{x = 7\}\mathit{skip}\{\mathit{odd}\ x\}$;
   (b) $\{x > 60\}x := x \times 2\{x > 100\}$;
   (c) $\{x > 40\}x := x \times 2\{x > 100\}$;
   (d) $\{\mathit{true}\}\mathbf{if}\ x \leqslant y \rightarrow y := y - x \mid x \geqslant y \rightarrow x := x - y\ \mathbf{fi}\{x \geqslant 0 \land y \geqslant 0\}$;
   (e) $\{\mathit{even}\ x \land \mathit{even}\ y\}\mathbf{if}\ x \leqslant y \rightarrow y := y - x \mid x \geqslant y \rightarrow x := x - y\ \mathbf{fi}\{\mathit{even}\ x \land \mathit{even}\ y\}$.

2. Is it always true that $\{\mathit{True}\}\ x := E\ \{x = E\}$? If you think the answer is yes, explain why. If your answer is no, give a counter example.

3. Verify:

$$\begin{array}{l} \{x = X \land y = Y\} \\ x := x \not\Leftrightarrow y \\ y := x \not\Leftrightarrow y \\ x := x \not\Leftrightarrow y \\ \{x = Y \land y = X\} \end{array}$$

   where $x$ and $y$ are boolean and $(\not\Leftrightarrow)$ is the "not equal" or "exclusive or" operator. In fact, the code above works for any $(\otimes)$ that satisfies the properties that for all $a$, $b$, and $c$:

$$\begin{array}{ll} \text{associative}: & a \otimes (b \otimes c) = (a \otimes b) \otimes c\ , \\ \text{unipotent}: & a \otimes a = 1\ , \end{array}$$

   where 1 is the unit of $(\otimes)$, that is, $1 \otimes b = b = b \otimes 1$.

4. Verify the following program:

$$\begin{array}{l} \mathbf{var}\ r, b : \mathit{Int} \\ \{0 \leqslant r < 2 \times b\} \\ \mathbf{if}\ b \leqslant r \rightarrow r := r - b \\ \mid\ r < b \rightarrow \mathit{skip} \\ \mathbf{fi} \\ \{0 \leqslant r < b\} \end{array}$$

5. Verify:

$$
\begin{array}{l}
\textbf{var } x, y : \mathit{Int} \\
\{\mathit{True}\} \\
x, y := x \times x, y \times y \\
\textbf{if } x \geqslant y \rightarrow x := x - y \\
\mid\ y \geqslant x \rightarrow y := y - x \\
\textbf{fi} \\
\{x \geqslant 0 \wedge y \geqslant 0\}\ \ .
\end{array}
$$

6. Verify:

$$
\begin{array}{l}
\textbf{var } a, b : \mathit{Bool} \\
\{\mathit{True}\} \\
\textbf{if } \neg\, a \vee b \rightarrow a := \neg\, a \\
\mid\ a \vee \neg\, b \rightarrow b := \neg\, b \\
\textbf{fi} \\
\{a \vee b\}\ \ .
\end{array}
$$

7. Assuming that $x$, $y$, and $z$ are integers, prove the following
   (a) $\{\mathit{True}\}$ **if** $x \geqslant 1 \rightarrow x := x + 1 \mid x \leqslant 1 \rightarrow x := x - 1$ **fi** $\{x \neq 1\}$.
   (b) $\{\mathit{True}\}$ **if** $x \geqslant y \rightarrow \mathit{skip} \mid y \geqslant x \rightarrow x, y := y, x$ **fi** $\{x \geqslant y\}$.
   (c) $\{x = 0\}$ **if** $\mathit{True} \rightarrow x := 1 \mid \mathit{True} \rightarrow x := -1\ \{x = 1 \vee x = -1\}$.
   (d) $\{A = x \times y + z\}$ **if** $\mathit{even}\ x \rightarrow x, y := x\ /\ 2, y \times 2 \mid \mathit{True} \rightarrow y, z := y - 1, z + x\ \{A = x \times y + z\}$.
   (e) $\{x \times y = 0 \wedge y \leqslant x\}$ **if** $y < 0 \rightarrow y := -y \mid y = 0 \rightarrow x := -1\ \{x < y\}$.

### Weakest Precondition of Simple Statements

8. Given below is a list of statements and predicates. What are the weakest precondition for the predicates to be true after the statement?
   (a) $x := x \times 2$, $x > 100$;
   (b) $x := x \times 2$, $\mathit{even}\ x$;
   (c) $x := x \times 2$, $x > 100 \wedge \mathit{even}\ x$;
   (d) $x := x \times 2$, $\mathit{odd}\ x$.
   (e) $\mathit{skip}$, $\mathit{odd}\ x$.
9. Determine the weakest $P$ that satisfies
   (a) $\{P\}\ x := x + 1; x := x + 1\ \{x \geqslant 0\}$.
   (b) $\{P\}\ x := x + y; y := 2 \times x\ \{y \geqslant 0\}$.
   (c) $\{P\}\ x := y; y := x\ \{x = A \wedge y = B\}$.
   (d) $\{P\}\ x := E; x := E\ \{x = E\}$.
10. What is the weakest $P$ such that the following holds?

$$
\begin{array}{l}
\textbf{var } x : \mathit{Int} \\
\{P\} \\
x := x + 1 \\
\textbf{if } x > 0 \rightarrow x := x + 1 \\
\mid\ x < 0 \rightarrow x := x + 2 \\
\mid\ x = 1 \rightarrow \mathit{skip} \\
\textbf{fi} \\
\{x \geqslant 1\}\ \ .
\end{array}
$$

11. Two programs $S_0$ and $S_1$ are equivalent if, for all $Q$, $wp\ S_0\ Q = wp\ S_1\ Q$. Show that the two following programs are equivalent.

$$\begin{array}{l}\mathbf{if}\ B_0 \rightarrow S_0 \mid B_1 \rightarrow S_1\ \mathbf{fi}; S \\ \mathbf{if}\ B_0 \rightarrow S_0; S \mid B_1 \rightarrow S_1; S\ \mathbf{fi}\end{array}$$

12. Consider the two programs:

$$\begin{array}{l}\mathsf{IF}_0 = \mathbf{if}\ B_0 \rightarrow S_0 \mid B_1 \rightarrow S_1\ \mathbf{fi}\ \ , \\ \mathsf{IF}_1 = \mathbf{if}\ B_0 \rightarrow S_0 \mid B_1 \wedge \neg B_0 \rightarrow S_1\ \mathbf{fi}\ \ .\end{array}$$

Show that for all $Q$, $wp\ \mathsf{IF}_0\ Q \Rightarrow wp\ \mathsf{IF}_1\ Q$.

**Properties of Weakest Precondition**

13. Prove that $(wp\ S\ Q_0 \vee wp\ S\ Q_1) \Rightarrow wp\ S\ (Q_0 \vee Q_1)$.

14. Recall the definition of Hoare triple in terms of $wp$:

$$\{P\}\,S\,\{Q\} \ = \ P \Rightarrow wp\ S\ Q \ \ .$$

Prove that

   1. $(\{P\}\,S\,\{Q\} \wedge (P_0 \Rightarrow P)) \Rightarrow \{P_0\}\,S\,\{Q\}$.
   2. $\{P\}\,S\,\{Q\} \wedge \{P\}\,S\,\{R\} \equiv \{P\}\,S\,\{Q \wedge R\}$.

15. Recall the weakest precondition of **if**:

$$wp\ (\mathbf{if}\ B_0 \rightarrow S_0 \mid B_1 \rightarrow S_1\ \mathbf{fi})\ Q \ = \ (B_0 \Rightarrow wp\ S_0\ Q) \wedge (B_1 \Rightarrow wp\ S_1\ Q) \wedge (B_0 \vee B_1) \ \ .$$

Prove that

$$\begin{array}{l}\{P\}\,\mathbf{if}\ B_0 \rightarrow S_0 \mid B_1 \rightarrow S_1\ \mathbf{fi}\,\{Q\} \ \equiv \\ \quad \{P \wedge B_0\}\,S\,\{Q\} \ \wedge \ \{P \wedge B_1\}\,S\,\{Q\} \ \wedge \ (P \Rightarrow (B_0 \vee B_1)) \ \ .\end{array}$$

**Note**: having proved so shows that the way we annotate **if** is correct:

$$\begin{array}{l}\{P\} \\ \mathbf{if}\ B_0 \rightarrow \{P \wedge B_0\}\,S_0\,\{Q\} \\ \ \mid\ B_1 \rightarrow \{P \wedge B_1\}\,S_1\,\{Q\} \\ \mathbf{fi} \\ \{Q\} \ \ .\end{array}$$

16. Recall that $wp\ S\ Q$ stands for "the weakest precondition for program $S$ to terminate in a state satisfying $Q$". What programs $S$, if any, satisfy each of the following conditions?

   1. $wp\ S\ \mathit{True} = \mathit{True}$.
   2. $wp\ S\ \mathit{True} = \mathit{False}$.
   3. $wp\ S\ \mathit{False} = \mathit{True}$.
   4. $wp\ S\ \mathit{False} = \mathit{False}$.