Programming Languages: Imperative Program Construction
Practicals 1: Non-Looping Constructs and Weakest Precondition

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Guarded Command Language Basics

1. Which of the following Hoare triples hold?
   (a) $\{x = 7\} skip \{odd\ x\}$;
   (b) $\{x > 60\} x := x \times 2 \{x > 100\}$;
   (c) $\{x > 40\} x := x \times 2 \{x > 100\}$;
   (d) $\{true\} \textbf{if}\ x \leqslant y \rightarrow y := y - x \mid x \geqslant y \rightarrow x := x - y\ \textbf{fi} \{x \geqslant 0 \wedge y \geqslant 0\}$;
   (e) $\{even\ x \wedge even\ y\} \textbf{if}\ x \leqslant y \rightarrow y := y - x \mid x \geqslant y \rightarrow x := x - y\ \textbf{fi} \{even\ x \wedge even\ y\}$.

Solution: As the first exercise I expect merely that you answer by informal reasoning. What follows is the more formal approach which you will learn later.

(a) The Hoare triple holds because:

$$\begin{aligned} & wp\ skip\ (odd\ x) \\ \equiv \quad & \{ \text{ definition of } wp\ \} \\ & odd\ x \\ \Leftarrow\ & x = 7 \ . \end{aligned}$$

(b) The Hoare triple holds because:

$$\begin{aligned} & wp\ (x := x \times 2)\ (x > 100) \\ \equiv \quad & \{ \text{ definition of } wp\ \} \\ & x \times 2 > 100 \\ \Leftarrow\ & x > 60 \ . \end{aligned}$$

(c) The Hoare triple does not hold because:

$$\begin{aligned} & wp\ (x := x \times 2)\ (x > 100) \\ \equiv\ & x \times 2 > 100 \\ \not\Leftarrow\ & x > 40 \ . \end{aligned}$$

(d) The annotated **if** statement is

$$\begin{array}{l} \{True\} \\ \textbf{if}\ x \leqslant y \rightarrow \{x \leqslant y\}\ y := y - x\ \{x \geqslant 0 \wedge y \geqslant 0\} \\ \quad x \geqslant y \rightarrow \{x \geqslant y\}\ x := x - y\ \{x \geqslant 0 \wedge y \geqslant 0\} \\ \textbf{fi} \\ \{x \geqslant 0 \wedge y \geqslant 0\} \ . \end{array}$$

That $x \leqslant y \lor x \geqslant y$ certainly holds. For the Hoare triple in the first branch we reason:

$$
\begin{aligned}
& (x \geqslant 0 \land y \geqslant 0)[y \backslash y - x] \\
\equiv\ & x \geqslant 0 \land y - x \geqslant 0 \\
\equiv\ & x \geqslant 0 \land x \leqslant y \\
\not\Leftarrow\ & x \leqslant y \ .
\end{aligned}
$$

The situation with the other branch is similar. The bottom line is that the initial Hoare triple does *not* hold.

The initial Hoare triple would be true if the precondition were $x \geqslant 0 \land y \geqslant 0$.

(e) The annotated **if** statement is

$$
\begin{aligned}
& \{\mathit{even}\ x \land \mathit{even}\ y\} \\
& \textbf{if}\ x \leqslant y \rightarrow \{\mathit{even}\ x \land \mathit{even}\ y \land x \leqslant y\}\ y := y - x\ \{\mathit{even}\ x \land \mathit{even}\ y\} \\
& \quad x \geqslant y \rightarrow \{\mathit{even}\ x \land \mathit{even}\ y \land x \geqslant y\}\ x := x - y\ \{\mathit{even}\ x \land \mathit{even}\ y\} \\
& \textbf{fi} \\
& \{\mathit{even}\ x \land \mathit{even}\ y\} \ .
\end{aligned}
$$

That $x \leqslant y \lor x \geqslant y$ certainly holds. For the Hoare triple in the first branch we reason:

$$
\begin{aligned}
& (\mathit{even}\ x \land \mathit{even}\ y)[y \backslash y - x] \\
\equiv\ & \mathit{even}\ x \land \mathit{even}\ (y - x) \\
\equiv\ & \mathit{even}\ x \land \mathit{even}\ y \\
\Leftarrow\ & \mathit{even}\ x \land \mathit{even}\ y \land x \leqslant y \ .
\end{aligned}
$$

The situation with the other branch is similar. The bottom line is that the initial Hoare triple does hold.

2. Is it always true that $\{\mathit{True}\}\ x := E\ \{x = E\}$? If you think the answer is yes, explain why. If your answer is no, give a counter example.

Solution: No. For a counterexample, let E be $x + 1$.

When do we do have the property that $\{\mathit{True}\}\ x := E\ \{x = E\}$? Since $(x = E)[x \backslash E] \equiv (E = E\ [x \backslash E])$, the Hoare triple holds if and only if $E = E\ [x \backslash E]$. Examples of such E include those that do not contain x, or those that are idempotent funtions on x, for example $E = 0 \uparrow x$.

The actual forward rule for assignment (due to Floyd) is:

$$\{P\}\ x := E\ \{(\exists\ x_0 :: x = E\ [x \backslash x_0] \land P\ [x \backslash x_0])\} \ ,$$

where x_0 is a fresh name.

3. Verify:

$$
\begin{aligned}
& \{x = X \land y = Y\} \\
& x := x \not\Leftrightarrow y \\
& y := x \not\Leftrightarrow y \\
& x := x \not\Leftrightarrow y \\
& \{x = Y \land y = X\}
\end{aligned}
$$

where x and y are boolean and $(\not\Leftrightarrow)$ is the “not equal” or “exclusive or” operator. In fact, the code above works for any $(\otimes)$ that satisfies the properties that for all a, b, and c:

$$\begin{array}{ll} \text{associative}: & a \otimes (b \otimes c) = (a \otimes b) \otimes c \ , \\ \text{unipotent}: & a \otimes a = 1 \ , \end{array}$$

where 1 is the unit of $(\otimes)$, that is, $1 \otimes b = b = b \otimes 1$.

Solution: The annotated program is:

$$\begin{array}{l} \{x = X \wedge y = Y, \mathrm{Pf}_2\} \\ x := x \otimes y \\ \{y = Y \wedge x \otimes y = X, \mathrm{Pf}_1\} \\ y := x \otimes y \\ \{x \otimes y = Y \wedge y = X\} \\ x := x \otimes y \\ \{x = Y \wedge y = X\} \ . \end{array}$$

Pf_1:

$$\begin{array}{ll} & (x \otimes y = Y \wedge y = X)\,[x \otimes y \,/\, y] \\ \equiv & x \otimes (x \otimes y) = Y \wedge x \otimes y = X \\ \equiv & \quad \{ (\otimes) \text{ associative} \} \\ & (x \otimes x) \otimes y = Y \wedge x \otimes y = X \\ \equiv & \quad \{ \text{unipotence} \} \\ & 1 \otimes y = Y \wedge x \otimes y = X \\ \equiv & \quad \{ \text{identity} \} \\ & y = Y \wedge x \otimes y = X \ . \end{array}$$

Pf_2:

$$\begin{array}{ll} & (y = Y \wedge x \otimes y = X)\,[x \otimes y \,/\, x] \\ \equiv & y = Y \wedge (x \otimes y) \otimes y = X \\ \equiv & \quad \{ (\otimes) \text{ associative} \} \\ & y = Y \wedge x \otimes (y \otimes y) = X \\ \equiv & \quad \{ \text{unipotence} \} \\ & y = Y \wedge x \otimes 1 = X \\ \equiv & \quad \{ \text{identity} \} \\ & y = Y \wedge x = X \ . \end{array}$$

4. Verify the following program:

$$\begin{array}{l} \mathbf{var}\ r, b : \mathit{Int} \\ \{0 \leqslant r < 2 \times b\} \\ \mathbf{if}\ b \leqslant r \rightarrow r := r - b \\ \mid\ r < b \rightarrow \mathit{skip} \\ \mathbf{fi} \\ \{0 \leqslant r < b\} \end{array}$$

Solution: The annotated program is:

$$
\begin{array}{l}
\textbf{var } r, b : \mathit{Int} \\
\{0 \leqslant r < 2 \times b\} \\
\textbf{if } b \leqslant r \rightarrow \{0 \leqslant r < 2 \times b \wedge b \leqslant r\}\ r := r - b\ \{0 \leqslant r < b, \mathrm{Pf}_1\} \\
\mid\ r < b \rightarrow \{0 \leqslant r < 2 \times b \wedge r < b\}\ \mathit{skip}\ \{0 \leqslant r < b, \mathrm{Pf}_2\} \\
\textbf{fi} \\
\{0 \leqslant r < b, \mathrm{Pf}_3\}
\end{array}
$$

Pf_1. We reason:

$$
\begin{array}{ll}
 & (0 \leqslant r < b)\ [r \backslash r - b] \\
\equiv & 0 \leqslant r - b < b \\
\equiv & b \leqslant r < 2 \times b \\
\Leftarrow & 0 \leqslant r < 2 \times b \wedge b \leqslant r\ .
\end{array}
$$

Pf_2. Trivial.

Pf_3. Certainly any proposition implies $b \leqslant r \vee r < b$.

5. Verify:

$$
\begin{array}{l}
\textbf{var } x, y : \mathit{Int} \\
\{\mathit{True}\} \\
x, y := x \times x, y \times y \\
\textbf{if } x \geqslant y \rightarrow x := x - y \\
\mid\ y \geqslant x \rightarrow y := y - x \\
\textbf{fi} \\
\{x \geqslant 0 \wedge y \geqslant 0\}\ .
\end{array}
$$

Solution: For brevity we abbreviate $x \geqslant 0 \ \wedge\ y \geqslant 0$ to P. The fully annotated program could be:

$$
\begin{array}{l}
\{\mathit{True}\} \\
x, y := x \times x, y \times y \\
\{P, \mathrm{Pf}_4\} \\
\textbf{if } x \geqslant y \rightarrow \{x \geqslant y \wedge P\}\ x := x - y\ \{P, \mathrm{Pf}_1\} \\
\mid\ y \geqslant x \rightarrow \{y \geqslant x \wedge P\}\ y := y - x\ \{P, \mathrm{Pf}_2\} \\
\textbf{fi} \\
\{P, \mathrm{Pf}_3\}\ .
\end{array}
$$

To verify the **if** branching, we check that

Pf_1. $\{x \geqslant y \ \wedge\ P\}\ x := x - y\ \{P\}$. The Hoare triple is valid because

$$
\begin{array}{ll}
 & (x \geqslant 0 \ \wedge\ y \geqslant 0)[x \backslash x - y] \\
\Leftrightarrow & x - y \geqslant 0 \ \wedge\ y \geqslant 0 \\
\Leftrightarrow & x \geqslant y \wedge\ y \geqslant 0 \\
\Leftarrow & x \geqslant y \ \wedge\ x \geqslant 0 \ \wedge\ y \geqslant 0.
\end{array}
$$

Pf_2. $\{y \geqslant x \ \wedge\ P\}\ y := y - x\ \{P\}$. Omitted.

Pf_3. And indeed $x \geqslant y \ \vee\ y \geqslant x$ always holds, thus $P \Rightarrow x \geqslant y \ \vee\ y \geqslant x$.

Do not forget that we have yet to verify $\{true\}\ x, y := x \times x, y \times y\ \{P\}$, which is not difficult either:

Pf_4.

$$
\begin{aligned}
& (x \geqslant 0 \ \wedge\ y \geqslant 0)[x, y \backslash x \times x, y \times y] \\
\Leftrightarrow\ & x \times x \geqslant 0 \ \wedge\ y \times y \geqslant 0 \\
\Leftrightarrow\ & true.
\end{aligned}
$$

6. Verify:

$$
\begin{aligned}
& \textbf{var } a, b : Bool \\
& \{True\} \\
& \textbf{if } \neg a \vee b \rightarrow a := \neg a \\
& \mid a \vee \neg b \rightarrow b := \neg b \\
& \textbf{fi} \\
& \{a \vee b\} \ .
\end{aligned}
$$

Solution:

$$
\begin{aligned}
& \textbf{var } a, b : Bool \\
& \{True\} \\
& \textbf{if } \neg a \vee b \rightarrow \{\neg a \vee b\}\ a := \neg a\ \{a \vee b, \text{Pf}_1\} \\
& \mid a \vee \neg b \rightarrow \{a \vee \neg b\}\ b := \neg b\ \{a \vee b, \text{Pf}_2\} \\
& \textbf{fi} \\
& \{a \vee b, \text{Pf}_3\} \ .
\end{aligned}
$$

Pf_1. To verify the first branch:

$$
\begin{aligned}
& (a \vee b)[a \backslash \neg a] \\
\equiv\ & \neg a \vee b.
\end{aligned}
$$

Pf_2. The other branch is similar.

Pf_3. Certainly $true \Rightarrow \neg a \vee b \vee a \vee \neg b$.

7. Assuming that x, y, and z are integers, prove the following
   (a) $\{True\}\ \textbf{if}\ x \geqslant 1 \rightarrow x := x + 1 \mid x \leqslant 1 \rightarrow x := x - 1\ \textbf{fi}\ \{x \neq 1\}$.
   (b) $\{True\}\ \textbf{if}\ x \geqslant y \rightarrow skip \mid y \geqslant x \rightarrow x, y := y, x\ \textbf{fi}\ \{x \geqslant y\}$.
   (c) $\{x = 0\}\ \textbf{if}\ True \rightarrow x := 1 \mid True \rightarrow x := -1\ \{x = 1 \vee x = -1\}$.
   (d) $\{A = x \times y + z\}\ \textbf{if}\ even\ x \rightarrow x, y := x \,/\, 2, y \times 2 \mid True \rightarrow y, z := y - 1, z + x\ \{A = x \times y + z\}$.

Solution: The annotated program is

$$\begin{array}{l}\{A = x \times y + z\}\\ \textbf{if } \mathit{even}\ x \rightarrow \{A = x \times y + z \wedge \mathit{even}\ x\}\ x, y := x \,/\, 2, y \times 2\ \{A = x \times y + z, \mathrm{Pf}_0\}\\ \mid\ \mathit{True} \quad \rightarrow \{A = x \times y + z\}\ y, z := y - 1, z + x\ \{A = x \times y + z, \mathrm{Pf}_1\}\\ \textbf{fi}\\ \{A = x \times y + z, \mathrm{Pf}_2\}\end{array}$$

Pf_0: We reason:

$$\begin{array}{ll} & (A = x \times y + z)[x, y \backslash x \,/\, 2, y \times 2]\\ \equiv & A = (x \,/\, 2) \times (y \times 2) + z\\ \Leftarrow & A = x \times y + z \wedge \mathit{even}\ x \ \ .\end{array}$$

Pf_2: We reason:

$$\begin{array}{ll} & (A = x \times y + z)[y, z \backslash y - 1, z + x]\\ \equiv & A = x \times (y - 1) + (z + x)\\ \Leftarrow & A = x \times y + z \ \ .\end{array}$$

Pf_2: Certainly $P \Rightarrow Q \wedge \mathit{True}$ for any P and Q.

(e) $\{x \times y = 0 \wedge y \leqslant x\}$ **if** $y < 0 \rightarrow y := -y \mid y = 0 \rightarrow x := -1\ \{x < y\}$.

Solution: The annotated program is

$$\begin{array}{l}\{x \times y = 0 \wedge y \leqslant x\}\\ \textbf{if } y < 0 \rightarrow \{x \times y = 0 \wedge y \leqslant x \wedge y < 0\}\ y := -y\ \{x < y, \mathrm{Pf}_0\}\\ \mid\ y = 0 \rightarrow \{x \times y = 0 \wedge y \leqslant x \wedge y = 0\}\ x := -1\ \{x < y, \mathrm{Pf}_1\}\\ \textbf{fi}\\ \{x < y, \mathrm{Pf}_2\}\end{array}$$

Pf_0: Note that $x \times y = 0$ equivals $x = 0 \vee y = 0$. Therefore

$$\begin{array}{ll} & x \times y = 0 \wedge y \leqslant x \wedge y < 0\\ \equiv & (x = 0 \vee y = 0) \wedge y \leqslant x \wedge y < 0\\ \equiv & \quad \{ \text{ distributivity } \}\\ & (x = 0 \wedge y \leqslant x \wedge y < 0) \vee (y = 0 \wedge y \leqslant x \wedge y < 0)\\ \equiv & \quad \{ \text{ since } (y = 0 \wedge y \leqslant x \wedge y < 0) \equiv \mathit{False} \ \}\\ & x = 0 \wedge y \leqslant x \wedge y < 0\\ \equiv & x = 0 \wedge y < 0 \ \ .\end{array}$$

To prove the Hoare triple we reason:

$$\begin{array}{ll} & (x < y)[y \backslash -y]\\ \equiv & x < -y\\ \Leftarrow & x = 0 \wedge y < 0 \ \ .\end{array}$$

Pf_1: We reason:

$$\begin{array}{ll} & (x < y)[x \backslash -1]\\ \equiv & -1 < y\\ \Leftarrow & x \times y = 0 \wedge y \leqslant x \wedge y = 0 \ \ .\end{array}$$

Pf$_2$: We reason:

$$
\begin{aligned}
& x \times y = 0 \wedge y \leqslant x \\
\equiv\ & (x = 0 \vee y = 0) \wedge y \leqslant x \\
\equiv\ & \quad \{ \text{ distributivity } \} \\
& (x = 0 \wedge y \leqslant x) \vee (y = 0 \wedge y \leqslant x) \\
\Rightarrow\ & y < 0 \vee y = 0 \ .
\end{aligned}
$$

Weakest Precondition of Simple Statements

8. Given below is a list of statements and predicates. What are the weakest precondition for the predicates to be true after the statement?
   (a) $x := x \times 2$, $x > 100$;
   (b) $x := x \times 2$, $\mathit{even}\ x$;
   (c) $x := x \times 2$, $x > 100 \wedge \mathit{even}\ x$;
   (d) $x := x \times 2$, $\mathit{odd}\ x$.
   (e) skip, $\mathit{odd}\ x$.

Solution:

(a) $x \times 2 > 100$, that is, $x > 50$.

(b) $\mathit{even}\ (x \times 2)$, which simplifies to *True*.

(c) $x \times 2 > 100 \wedge \mathit{even}\ (x \times 2)$, that is, $x > 50$.

(d) $\mathit{odd}\ (x \times 2)$, that is, *False*.

(e) $\mathit{odd}\ x$.

9. Determine the weakest P that satisfies
   (a) $\{P\}\ x := x + 1; x := x + 1\ \{x \geqslant 0\}$.
   (b) $\{P\}\ x := x + y; y := 2 \times x\ \{y \geqslant 0\}$.
   (c) $\{P\}\ x := y; y := x\ \{x = A \wedge y = B\}$.
   (d) $\{P\}\ x := E; x := E\ \{x = E\}$.

Solution:

(a)

$$
\begin{aligned}
& \mathit{wp}\ (x := x + 1; x := x + 1)\ (x \geqslant 0) \\
=\ & \mathit{wp}\ (x := x + 1)\ (\mathit{wp}\ (x := x + 1)\ (x \geqslant 0)) \\
=\ & \mathit{wp}\ (x := x + 1)\ (x + 1 \geqslant 0) \\
=\ & (x + 1) + 1 \geqslant 0 \\
=\ & x \geqslant -2 \ .
\end{aligned}
$$

(b)

$$\begin{aligned} & wp\ (x := x + y; y := 2 \times x)\ (y \geqslant 0) \\ = {} & wp\ (x := x + y)\ (wp\ (y := 2 \times x)\ (y \geqslant 0)) \\ = {} & wp\ (x := x + y)\ (2 \times x \geqslant 0) \\ = {} & 2 \times (x + y) \geqslant 0 \ . \end{aligned}$$

(c)

$$\begin{aligned} & wp\ (x := y; y := x)\ (x = A \wedge y = B) \\ \equiv {} & wp\ (x := y)\ (wp\ (y := x)\ (x = A \wedge y = B)) \\ \equiv {} & wp\ (x := y)\ (x = A \wedge x = B) \\ \equiv {} & y = A \wedge y = B \\ \equiv {} & y = A = B \ . \end{aligned}$$

(d)

$$\begin{aligned} & wp\ (x := E; x := E)\ (x = E) \\ \equiv {} & wp\ (x := E)\ (wp\ (x := E)\ (x = E)) \\ \equiv {} & wp\ (x := E)\ ((x = E)[x \backslash E]) \\ \equiv {} & wp\ (x := E)\ (E = E[x \backslash E]) \\ \equiv {} & (E = E[x \backslash E])[x \backslash E] \\ \equiv {} & E[x \backslash E] = (E[x \backslash E])[x \backslash E] \ . \end{aligned}$$

The equation certainly does not hold in general. One example where it does hold is $E = (-x) \uparrow 0$, for which we have:

$$\begin{aligned} & E[x \backslash E] \\ = {} & (-((-x) \uparrow 0)) \uparrow 0 \\ = {} & (x \downarrow 0) \uparrow 0 \\ = {} & 0 \\ = {} & (-0) \uparrow 0 \\ = {} & (-((-((-x) \uparrow 0)) \uparrow 0)) \uparrow 0 \\ = {} & (E[x \backslash E])[x \backslash E] \ . \end{aligned}$$

Let me know if you have a more interesting E.

10. What is the weakest P such that the following holds?

    var $x : Int$
    $\{P\}$
    $x := x + 1$
    if $x > 0 \rightarrow x := x + 1$
    $\mid\ x < 0 \rightarrow x := x + 2$
    $\mid\ x = 1 \rightarrow skip$
    fi
    $\{x \geqslant 1\}$.

Solution: Denote the **if** statement by IF. The aim is to compute $wp\ (x := x + 1; \mathrm{IF})\ (x \geqslant 1)$.

Recall the definition of wp for **if**. We have

$$\begin{aligned} wp\ \mathrm{IF}\ (x \geqslant 1) = {} & (x > 0 \Rightarrow wp\ (x := x + 1)\ (x \geqslant 1)) \wedge \\ & (x < 0 \Rightarrow wp\ (x := x + 2)\ (x \geqslant 1)) \wedge \\ & (x = 1 \Rightarrow wp\ skip\ (x \geqslant 1)) \wedge \\ & (x > 0 \vee x < 0 \vee x = 1) \ . \end{aligned}$$

We calculate the four conjuncts separately:

- $$\begin{aligned} & x > 0 \Rightarrow wp\ (x := x+1)\ (x \geqslant 1) \\ \equiv\ & x > 0 \Rightarrow x + 1 \geqslant 1 \\ \equiv\ & x > 0 \Rightarrow x \geqslant 0 \\ \equiv\ & True\ . \end{aligned}$$

- $$\begin{aligned} & x < 0 \Rightarrow wp\ (x := x+2)\ (x \geqslant 1) \\ \equiv\ & x < 0 \Rightarrow x + 2 \geqslant 1 \\ \equiv\ & x < 0 \Rightarrow x \geqslant -1 \\ \equiv\ & \quad \{ (P \Rightarrow Q) = (\neg P \vee Q) \} \\ & x \geqslant 0 \vee x \geqslant -1 \\ \equiv\ & x \geqslant -1\ . \end{aligned}$$

- $$\begin{aligned} & x = 1 \Rightarrow wp\ skip\ (x \geqslant 1) \\ \equiv\ & x = 1 \Rightarrow x \geqslant 1 \\ \equiv\ & True\ . \end{aligned}$$

- Furthermore, $x > 0 \vee x < 0 \vee x = 1$ simplifies to $x \neq 0$.

Therefore,

$$\begin{aligned} & wp\ \mathrm{IF}\ (x \geqslant 1) \\ =\ & True \wedge x \geqslant -1 \wedge True \wedge x \neq 0 \\ =\ & x \geqslant -1 \wedge x \neq 0\ . \end{aligned}$$

Finally, recall what we want to compute:

$$\begin{aligned} & wp\ (x := x + 1; \mathrm{IF})\ (x \geqslant 1) \\ =\ & wp\ (x := x+1)\ (wp\ \mathrm{IF}\ (x \geqslant 1)) \\ =\ & wp\ (x := x+1)\ (x \geqslant -1 \wedge x \neq 0) \\ =\ & x + 1 \geqslant -1 \wedge x + 1 \neq 0) \\ =\ & x \geqslant -2 \wedge x \neq -1\ . \end{aligned}$$

11. Two programs S_0 and S_1 are equivalent if, for all Q, $wp\ S_0\ Q = wp\ S_1\ Q$. Show that the two following programs are equivalent.

$\textbf{if}\ B_0 \rightarrow S_0 \mid B_1 \rightarrow S_1\ \textbf{fi}; S$
$\textbf{if}\ B_0 \rightarrow S_0; S \mid B_1 \rightarrow S_1; S\ \textbf{fi}$

Solution:

$$
\begin{aligned}
& wp\ (\textbf{if}\ B_0 \to S_0 \mid B_1 \to S_1\ \textbf{fi}; S)\ Q \\
= \quad & \{ \text{ definition of } wp \ \} \\
& wp\ (\textbf{if}\ B_0 \to S_0 \mid B_1 \to S_1\ \textbf{fi})\ (wp\ S\ Q) \\
= \quad & \{ \text{ definition of } wp \ \} \\
& (B_0 \Rightarrow wp\ S_0\ (wp\ S\ Q)) \land \\
& (B_1 \Rightarrow wp\ S_1\ (wp\ S\ Q)) \land (B_0 \lor B_1) \\
= \quad & \{ \text{ definition of } wp \ \} \\
& (B_0 \Rightarrow wp\ (S_0; S)\ Q) \land \\
& (B_1 \Rightarrow wp\ (S_1; S)\ Q) \land (B_0 \lor B_1) \\
= \quad & \{ \text{ definition of } wp \ \} \\
& wp\ (\textbf{if}\ B_0 \to S_0; S \mid B_1 \to S_1; S\ \textbf{fi})\ Q \ \ .
\end{aligned}
$$

12. Consider the two programs:

$$
\begin{aligned}
\mathsf{IF}_0 &= \textbf{if}\ B_0 \to S_0 \mid B_1 \to S_1\ \textbf{fi} \ \ , \\
\mathsf{IF}_1 &= \textbf{if}\ B_0 \to S_0 \mid B_1 \land \neg B_0 \to S_1\ \textbf{fi} \ \ .
\end{aligned}
$$

Show that for all Q, $wp\ \mathsf{IF}_0\ Q \Rightarrow wp\ \mathsf{IF}_1\ Q$.

Solution: Firstly, we show that $B_0 \lor (B_1 \land \neg B_0) = B_0 \lor B_1$.

$$
\begin{aligned}
& B_0 \lor (B_1 \land \neg B_0) \\
= \quad & \{ \text{ distributivity } \} \\
& (B_0 \lor B_1) \land (B_0 \lor \neg B_0) \\
= \ & (B_0 \lor B_1) \land \mathit{True} \\
= \ & B_0 \lor B_1 \ \ .
\end{aligned}
$$

Secondly, recall that

- conjunction is monotonic, that is, $(P_0 \land Q) \Rightarrow (P_1 \land Q)$ if $P_0 \Rightarrow P_1$;
- implication is anti-monotonic in its first argument, that is $(P_0 \Rightarrow Q) \Rightarrow (P_1 \Rightarrow Q)$ if $P_1 \Rightarrow P_0$.

Therefore we have

$$
\begin{aligned}
& wp\ (\textbf{if}\ B_0 \to S_0 \mid B_1 \to S_1\ \textbf{fi})\ Q \\
= \ & (B_0 \Rightarrow wp\ S_0\ Q) \land (B_1 \Rightarrow wp\ S_1\ Q) \land (B_0 \lor B_1) \\
= \quad & \{ \text{ since } B_0 \lor (B_1 \land \neg B_0) = B_0 \lor B_1 \ \} \\
& (B_0 \Rightarrow wp\ S_0\ Q) \land (B_1 \Rightarrow wp\ S_1\ Q) \land (B_0 \lor (B_1 \land \neg B_0)) \\
\Rightarrow \quad & \{ \text{ since } B_1 \land \neg B_0 \Rightarrow B_1 \text{, (anti-)monotonicity as discussed above. } \} \\
& (B_0 \Rightarrow wp\ S_0\ Q) \land (B_1 \land \neg B_0 \Rightarrow wp\ S_1\ Q) \land (B_0 \lor (B_1 \land \neg B_0)) \\
= \ & wp\ (\textbf{if}\ B_0 \to S_0 \mid B_1 \land \neg B_0 \to S_1\ \textbf{fi})\ Q \ \ .
\end{aligned}
$$

Properties of Weakest Precondition

13. Prove that $(wp\ S\ Q_0 \lor wp\ S\ Q_1) \Rightarrow wp\ S\ (Q_0 \lor Q_1)$.

Solution: Recall from propositional logic that $(P \lor Q) \Rightarrow R$ iff. $(P \Rightarrow R) \land (Q \Rightarrow R)$.

$$
\begin{array}{ll}
 & (wp\ S\ Q_0 \lor wp\ S\ Q_1) \Rightarrow wp\ S\ (Q_0 \lor Q_1) \\
\equiv & \quad \{ \text{said property above} \} \\
 & (wp\ S\ Q_0 \Rightarrow wp\ S\ (Q_0 \lor Q_1)) \land \\
 & (wp\ S\ Q_1 \Rightarrow wp\ S\ (Q_0 \lor Q_1)) \\
\Leftarrow & \quad \{ \text{Monotonicity} \} \\
 & (Q_0 \Rightarrow (Q_0 \lor Q_1)) \land (Q_1 \Rightarrow (Q_0 \lor Q_1)) \\
\equiv & \mathit{True} \ .
\end{array}
$$

14. Recall the definition of Hoare triple in terms of wp:

$$\{P\}\,S\,\{Q\} \ = \ P \Rightarrow wp\ S\ Q \ .$$

Prove that

1. $(\{P\}\,S\,\{Q\} \land (P_0 \Rightarrow P)) \Rightarrow \{P_0\}\,S\,\{Q\}$.
2. $\{P\}\,S\,\{Q\} \land \{P\}\,S\,\{R\} \equiv \{P\}\,S\,\{Q \land R\}$.

Solution:

1. We reason:

$$
\begin{array}{ll}
 & \{P_0\}\,S\,\{Q\} \\
\equiv & \quad \{ \text{definition of Hoare triple} \} \\
 & P_0 \Rightarrow wp\ S\ Q \\
\Leftarrow & \quad \{ \text{since } P_0 \Rightarrow P \} \\
 & P \Rightarrow wp\ S\ Q \\
\equiv & \quad \{ \text{definition of Hoare triple} \} \\
 & \{P\}\,S\,\{Q\} \ .
\end{array}
$$

2. We reason:

$$
\begin{array}{ll}
 & \{P\}\,S\,\{Q \land R\} \\
\equiv & \quad \{ \text{definition of Hoare triple} \} \\
 & P \Rightarrow wp\ S\ (Q \land R) \\
\equiv & \quad \{ \text{distributivity over conjunction} \} \\
 & P \Rightarrow (wp\ S\ Q \land wp\ S\ R) \\
\equiv & \quad \{ \text{since } (P \Rightarrow (X \land Y)) \equiv (P \Rightarrow X) \land (P \Rightarrow Y) \} \\
 & (P \Rightarrow wp\ S\ Q) \land (P \Rightarrow wp\ S\ R) \\
\equiv & \quad \{ \text{definition of Hoare triple} \} \\
 & \{P\}\,S\,\{Q\} \land \{P\}\,S\,\{R\} \ .
\end{array}
$$

15. Recall the weakest precondition of **if**:

$$wp\ (\mathbf{if}\ B_0 \rightarrow S_0 \mid B_1 \rightarrow S_1\ \mathbf{fi})\ Q \ = \ (B_0 \Rightarrow wp\ S_0\ Q) \land (B_1 \Rightarrow wp\ S_1\ Q) \land (B_0 \lor B_1) \ .$$

Prove that

$$\{P\}\,\textbf{if}\ B_0 \to S_0 \mid B_1 \to S_1\ \textbf{fi}\,\{Q\} \ \equiv$$
$$\{P \wedge B_0\}\, S\,\{Q\} \ \wedge\ \{P \wedge B_1\}\, S\,\{Q\} \ \wedge\ (P \Rightarrow (B_0 \vee B_1)) \ .$$

Note: having proved so shows that the way we annotate **if** is correct:

$$\begin{array}{l} \{P\} \\ \textbf{if}\ B_0 \to \{P \wedge B_0\}\, S_0\,\{Q\} \\ \mid\ B_1 \to \{P \wedge B_1\}\, S_1\,\{Q\} \\ \textbf{fi} \\ \{Q\} \ . \end{array}$$

Solution: We reason:

$$\begin{array}{ll} & \{P\}\,\textbf{if}\ B_0 \to S_0 \mid B_1 \to S_1\ \textbf{fi}\,\{Q\} \\ = & \quad \{ \text{definition of Hoare triple} \} \\ & P \Rightarrow wp\ (\textbf{if}\ B_0 \to S_0 \mid B_1 \to S_1\ \textbf{fi})\ Q \\ = & \quad \{ \text{definition of } wp \} \\ & P \Rightarrow ((B_0 \Rightarrow wp\ S_0\ Q) \wedge (B_1 \Rightarrow wp\ S_1\ Q) \wedge (B_0 \vee B_1)) \\ = & \quad \{ \text{since } (P \Rightarrow (Q \wedge R)) \equiv (P \Rightarrow Q) \wedge (P \Rightarrow R) \} \\ & (P \Rightarrow (B_0 \Rightarrow wp\ S_0\ Q)) \wedge \\ & (P \Rightarrow (B_1 \Rightarrow wp\ S_1\ Q)) \wedge \\ & (P \Rightarrow (B_0 \vee B_1)) \\ = & \quad \{ \text{since } (P \Rightarrow (Q \Rightarrow R)) \equiv ((P \wedge Q) \Rightarrow R) \} \\ & ((P \wedge B_0) \Rightarrow wp\ S_0\ Q) \wedge \\ & ((P \wedge B_1) \Rightarrow wp\ S_1\ Q) \wedge \\ & (P \Rightarrow (B_0 \vee B_1)) \\ = & \quad \{ \text{definition of Hoare triple} \} \\ & \{P \wedge B_0\}\, S_0\,\{Q\} \wedge \\ & \{P \wedge B_1\}\, S_1\,\{Q\} \wedge \\ & (P \Rightarrow (B_0 \vee B_1)) \ . \end{array}$$

16. Recall that $wp\ S\ Q$ stands for "the weakest precondition for program S to terminate in a state satisfying Q". What programs S, if any, satisfy each of the following conditions?

    1. $wp\ S\ True = True$.
    2. $wp\ S\ True = False$.
    3. $wp\ S\ False = True$.
    4. $wp\ S\ False = False$.

Solution:

1. $wp\ S\ True = True$: S is a program that always terminates.
2. $wp\ S\ True = False$: S is a program that never terminates.
3. $wp\ S\ False = True$: there is no such a program S.
4. $wp\ S\ False = False$: S can be any program.