Programming Languages: Imperative Program Construction
Practicals 2. Propositional Logic

Shin-Cheng Mu

Autumn Term, 2024

Prove each of the following properties using only axioms or theorems established before it (for example, prove (3.11) using only (1.?) and (3.1) - (3.10)).

Note that there are more than one ways to prove a property. You may discover a proof that is better than the one given in the solution.

1. Prove (3.9): $\neg(p \equiv q) \equiv \neg p \equiv q$.
2. Prove (3.12): $\neg\neg p \equiv p$.
3. Prove (3.13): $\neg \mathit{False} \equiv \mathit{True}$.
4. Prove (3.29): $p \vee \mathit{True} \equiv \mathit{True}$.
5. Prove (3.32): $p \vee q \equiv p \vee \neg q \equiv p$.
6. Prove (3.42): $p \wedge \neg p \equiv \mathit{False}$.
7. Prove (3.43a): $p \wedge (p \vee q) \equiv p$.
8. Prove (3.44a). $p \wedge (\neg p \vee q) \equiv p \wedge q$.
9. Prove (3.65): $p \wedge q \Rightarrow r \ \equiv \ p \Rightarrow (q \Rightarrow r)$.
10. Prove (3.66): $p \wedge (p \Rightarrow q) \equiv p \wedge q$.
11. Prove (3.67): $p \wedge (q \Rightarrow p) \equiv p$.
12. Prove (3.68): $p \vee (p \Rightarrow q) \equiv \mathit{True}$.
13. Prove (3.69): $p \vee (q \Rightarrow p) \equiv q \Rightarrow p$.
14. Prove (3.78): $(p \Rightarrow r) \wedge (q \Rightarrow r) \equiv (p \vee q \Rightarrow r)$.
15. Prove that $(p \Rightarrow q) \wedge (p \Rightarrow r) \equiv (p \Rightarrow q \wedge r)$.
16. Prove that $(r \Rightarrow)$ is monotonic with respect to implication. That is, $(p \Rightarrow q) \Rightarrow ((r \Rightarrow p) \Rightarrow (r \Rightarrow q))$.
17. Prove that $(\Rightarrow r)$ is anti-monotonic with respect to implication. That is, $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$.
18. Prove that conjunction is monotonic with respect to implication. That is, $(p \Rightarrow q) \Rightarrow ((p \wedge r) \Rightarrow (q \wedge r))$.
19. Prove (4.3) $(p \Rightarrow q) \Rightarrow (p \wedge r \Rightarrow q \wedge r)$. Start with the consequent, since it has more structure.
20. Prove (3.76d) $p \vee (q \wedge r) \ \Rightarrow \ p \vee q$ using inequality reasoning. Start with the antecedent, since it has more structure, and use distributivity.
21. Prove $(p \Rightarrow q) \wedge (r \Rightarrow s) \ \Rightarrow \ (p \vee r \Rightarrow q \vee s)$ using inequality reasoning. **Hint**: first remove the implication in the antecedent, distribute as much as possible, and use (3.76d) and an absorption theorem.

22. Prove (4.1) $p \Rightarrow (q \Rightarrow p)$ by assuming the antecedent.
23. Prove $(\neg p \Rightarrow q) \Rightarrow ((p \Rightarrow q) \Rightarrow q)$ by assuming the antecedent.
24. Prove $(p \Rightarrow p') \wedge (q \Rightarrow q') \Rightarrow (p \vee q \Rightarrow p' \vee q')$ by assuming the antecedent.