Programming Languages: Imperative Program Construction
Practicals 2. Propositional Logic

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Prove each of the following properties using only axioms or theorems established before it (for example, prove (3.11) using only (1.?) and (3.1) - (3.10)).

Note that there are more than one ways to prove a property. You may discover a proof that is better than the one given in the solution.

1. Prove (3.9): $\neg(p \equiv q) \equiv \neg p \equiv q$.

Solution:

$$
\begin{array}{ll}
& \neg(p \equiv q) \\
= & \{\ \text{definition of } \mathit{False} \text{ (3.15)}\ \} \\
& p \equiv q \equiv \mathit{False} \\
= & \{\ \text{symmetry (3.2) and associativity (3.1) of } \equiv\ \} \\
& p \equiv \mathit{False} \equiv q \\
= & \{\ \text{definition of } \mathit{False} \text{ (3.15)}\ \} \\
& \neg p \equiv q
\end{array}
$$

2. Prove (3.12): $\neg\neg p \equiv p$.

Solution:

$$
\begin{array}{ll}
& \neg\neg p \\
= & \{\ \text{definition of } \mathit{False} \text{ (3.5), associativity (3.1) and symmetry (3.2) of } \equiv\ \} \\
& \neg p \equiv \mathit{False} \\
= & \{\ \text{definition of } \mathit{False} \text{ (3.5)}\ \} \\
& p
\end{array}
$$

3. Prove (3.13): $\neg \mathit{False} \equiv \mathit{True}$.

Solution:

$$
\neg \mathit{False}
$$

$$
\begin{aligned}
= \quad & \{ \text{ definition of } \mathit{False} \text{ (3.5) } \} \\
& \mathit{False} \equiv \mathit{False} \\
= \quad & \{ \text{ identity of } \equiv \text{ (3.3) } \} \\
& \mathit{True}
\end{aligned}
$$

4. Prove (3.29): $p \vee \mathit{True} \equiv \mathit{True}$.

Solution:

$$
\begin{aligned}
& p \vee \mathit{True} \\
= \quad & \{ \text{ identity of } \equiv \text{ (3.3) } \} \\
& p \vee (p \equiv p) \\
= \quad & \{ \text{ distributivity (3.27) } \} \\
& p \vee p \equiv p \vee p \\
= \quad & \{ \text{ identity of } \equiv \text{ (3.3) } \} \\
& \mathit{True}
\end{aligned}
$$

5. Prove (3.32): $p \vee q \equiv p \vee \neg q \equiv p$.

Solution:

$$
\begin{aligned}
& p \vee q \equiv p \vee \neg q \\
= \quad & \{ \text{ distributivity (3.27) } \} \\
& p \vee (q \equiv \neg q) \\
= \quad & \{ \text{ definition of } \mathit{False} \text{ (3.15) } \} \\
& p \vee \mathit{False} \\
= \quad & \{ \text{ identity of } \vee \text{ (3.30) } \} \\
& p
\end{aligned}
$$

6. Prove (3.42): $p \wedge \neg p \equiv \mathit{False}$.

Solution:

$$
\begin{aligned}
& p \wedge \neg p \\
= \quad & \{ \text{ golden rule (3.35) } \} \\
& p \equiv \neg p \equiv p \vee \neg p \\
= \quad & \{ \text{ excluded middle (3.28) } \} \\
& p \equiv \neg p \equiv \mathit{True} \\
= \quad & \{ \text{ identity of } \mathit{True} \text{ (3.3) } \}
\end{aligned}
$$

$$\begin{aligned}
& p \equiv \neg p \\
= \quad & \{ \text{ definition of } \textit{False} \text{ (3.15) } \} \\
& \textit{False}
\end{aligned}$$

Another proof:

$$\begin{aligned}
& p \wedge \neg p \equiv \textit{False} \\
= \quad & \{ \text{ definition of } \textit{False} \text{ (3.15) } \} \\
& p \wedge \neg p \equiv p \equiv \neg p \\
= \quad & \{ \text{ golden rule (3.35) } \} \\
& p \vee \neg p \\
= \quad & \{ \text{ excluded middle (3.28) } \} \\
& \textit{True}
\end{aligned}$$

7. Prove (3.43a): $p \wedge (p \vee q) \equiv p$.

Solution:

$$\begin{aligned}
& p \wedge (p \vee q) \\
= \quad & \{ \text{ golden rule (3.35) } \} \\
& p \equiv p \vee q \equiv p \vee p \vee q \\
= \quad & \{ \text{ idempotency of } \vee \text{ (3.26) } \} \\
& p \equiv p \vee q \equiv p \vee q \\
= \quad & \{ \text{ identity of } \equiv \text{ (3.3) } \} \\
& p \equiv \textit{True} \\
= \quad & \{ \text{ identity of } \equiv \text{ (3.3) } \} \\
& p
\end{aligned}$$

8. Prove (3.44a). $p \wedge (\neg p \vee q) \equiv p \wedge q$.

Solution:

$$\begin{aligned}
& p \wedge (\neg p \vee q) \\
= \quad & \{ \text{ golden rule (3.35) } \} \\
& p \equiv \neg p \vee q \equiv p \vee \neg p \vee q \\
= \quad & \{ \text{ excluded middle (3.28) } \} \\
& p \equiv \neg p \vee q \equiv \textit{True} \vee q \\
= \quad & \{ \text{ zero of } \vee \text{ (3.29) and identity of } \equiv \text{ (3.3) } \} \\
& p \equiv \neg p \vee q \\
= \quad & \{ \text{ (3.32), with } p, q := q, p \} \\
& p \equiv q \equiv p \vee q
\end{aligned}$$

$$
\begin{aligned}
= \quad & \{ \text{ golden rule (3.35) } \} \\
& p \wedge q
\end{aligned}
$$

9. Prove (3.65): $p \wedge q \Rightarrow r \;\equiv\; p \Rightarrow (q \Rightarrow r)$.

Solution:

$$
\begin{aligned}
& p \Rightarrow (q \Rightarrow r) \\
= \quad & \{ \text{ definition of } \Rightarrow \text{ (3.60) } \} \\
& p \wedge (q \Rightarrow r) \equiv p \\
= \quad & \{ \text{ definition of } \Rightarrow \text{ (3.60) } \} \\
& p \wedge (q \wedge r \equiv q) \equiv p \\
= \quad & \{ \text{ (3.49) } \} \\
& p \wedge q \wedge r \equiv p \wedge q \equiv p \equiv p \\
= \quad & \{ \text{ identity of } \equiv \text{ (3.3) } \} \\
& p \wedge q \wedge r \equiv p \wedge q \\
= \quad & \{ \text{ definition of } \Rightarrow \text{ (3.60) } \} \\
& (p \wedge q) \Rightarrow r
\end{aligned}
$$

10. Prove (3.66): $p \wedge (p \Rightarrow q) \equiv p \wedge q$.

Solution:

$$
\begin{aligned}
& p \wedge (p \Rightarrow q) \\
= \quad & \{ \text{ definition of } \Rightarrow \text{ (3.60) } \} \\
& p \wedge (p \wedge q \equiv p) \\
= \quad & \{ \text{ (3.49) and idempotency of } \wedge \text{ (3.38) } \} \\
& p \wedge q \equiv p \equiv p \\
= \quad & \{ \text{ identity of } \equiv \text{ (3.3) } \} \\
& p \wedge q
\end{aligned}
$$

11. Prove (3.67): $p \wedge (q \Rightarrow p) \equiv p$.

Solution:

$$
\begin{aligned}
& p \wedge (q \Rightarrow p) \\
= \quad & \{ \text{ definition of } \Rightarrow \text{ (3.60) } \} \\
& p \wedge (q \wedge p \equiv q) \\
= \quad & \{ \text{ (3.49) and idempotency of } \wedge \text{ (3.38) } \}
\end{aligned}
$$

$$p \wedge q \equiv p \wedge q \equiv p$$
$$= \quad \{ \text{ identity of } \equiv (3.3) \}$$
$$p$$

12. Prove (3.68): $p \vee (p \Rightarrow q) \equiv \mathit{True}$.

Solution:

$$p \vee (p \Rightarrow q)$$
$$= \quad \{ \text{ definition of } \Rightarrow (3.57) \}$$
$$p \vee (p \vee q \equiv q)$$
$$= \quad \{ \text{ distributvity (3.27) and idempotency (3.26) } \}$$
$$p \vee q \equiv p \vee q$$
$$= \quad \{ \text{ identity of } \equiv (3.3) \}$$
$$\mathit{True}$$

Another proof:

$$p \vee (p \Rightarrow q)$$
$$= \quad \{ \text{ definition of } \Rightarrow (3.59) \}$$
$$p \vee \neg p \vee q$$
$$= \quad \{ \text{ excluded middle (3.28) and zero of } \vee (3.29) \}$$
$$\mathit{True}$$

13. Prove (3.69): $p \vee (q \Rightarrow p) \equiv q \Rightarrow p$.

Solution:

$$p \vee (q \Rightarrow p)$$
$$= \quad \{ \text{ definition of } \Rightarrow (3.59) \}$$
$$p \vee \neg q \vee p$$
$$= \quad \{ \text{ idempotency of } \vee (3.26) \}$$
$$p \vee \neg q$$
$$= \quad \{ \text{ definition of } \Rightarrow (3.59) \}$$
$$q \Rightarrow p$$

14. Prove (3.78): $(p \Rightarrow r) \wedge (q \Rightarrow r) \equiv (p \vee q \Rightarrow r)$.

Solution:

$$
\begin{aligned}
& p \vee q \Rightarrow r \\
= \quad & \{ \text{ definition of } \Rightarrow (3.59) \; \} \\
& \neg(p \vee q) \vee r \\
= \quad & \{ \text{ de Morgan } (3.47) \; \} \\
& (\neg p \wedge \neg q) \vee r \\
= \quad & \{ \text{ distributivity } (3.45) \; \} \\
& (\neg p \vee r) \wedge (\neg q \vee r) \\
= \quad & \{ \text{ definition of } \Rightarrow (3.59) \; \} \\
& (p \Rightarrow r) \wedge (q \Rightarrow r)
\end{aligned}
$$

15. Prove that $(p \Rightarrow q) \wedge (p \Rightarrow r) \equiv (p \Rightarrow q \wedge r)$.

Solution:

$$
\begin{aligned}
& (p \Rightarrow q) \wedge (p \Rightarrow r) \\
= \quad & \{ \text{ definition of } \Rightarrow (3.59) \; \} \\
& (\neg p \vee q) \wedge (\neg p \vee r) \\
= \quad & \{ \text{ distributivity } (3.45) \; \} \\
& \neg p \vee (q \wedge r) \\
= \quad & \{ \text{ definition of } \Rightarrow (3.59) \; \} \\
& p \Rightarrow q \wedge r
\end{aligned}
$$

16. Prove that $(r \Rightarrow)$ is monotonic with respect to implication. That is, $(p \Rightarrow q) \Rightarrow ((r \Rightarrow p) \Rightarrow (r \Rightarrow q))$.

Solution:

$$
\begin{aligned}
& (p \Rightarrow q) \Rightarrow ((r \Rightarrow p) \Rightarrow (r \Rightarrow q)) \\
= \quad & \{ \text{ shunting } (3.65) \; \} \\
& ((p \Rightarrow q) \wedge (r \Rightarrow p)) \Rightarrow (r \Rightarrow q) \\
= \quad & \{ \text{ shunting } (3.65) \} \\
& ((p \Rightarrow q) \wedge (r \Rightarrow p) \wedge r) \Rightarrow q \\
= \quad & \{ \; (3.66) \; \} \\
& ((p \Rightarrow q) \wedge p \wedge r) \Rightarrow q \\
= \quad & \{ \; (3.66) \; \} \\
& (q \wedge p \wedge r) \Rightarrow q \\
= \quad & \{ \text{ weakening } (3.76b) \; \} \\
& \mathit{True}
\end{aligned}
$$

17. Prove that $(\Rightarrow r)$ is anti-monotonic with respect to implication. That is, $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$.

Solution:

$$
\begin{array}{ll}
 & (p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r)) \\
= & \{ \text{ shunting (3.65) } \} \\
 & ((p \Rightarrow q) \wedge (q \Rightarrow r)) \Rightarrow (p \Rightarrow r) \\
= & \{ \text{ shunting (3.65) } \} \\
 & ((p \Rightarrow q) \wedge (q \Rightarrow r) \wedge p) \Rightarrow r \\
= & \{ \text{ (3.66) } \} \\
 & (p \wedge q \wedge (q \Rightarrow r)) \Rightarrow r \\
= & \{ \text{ (3.66) } \} \\
 & (p \wedge q \wedge r) \Rightarrow r \\
= & \{ \text{ weakening (3.76b) } \} \\
 & \mathit{True}
\end{array}
$$

18. Prove that conjunction is monotonic with respect to implication. That is, $(p \Rightarrow q) \Rightarrow ((p \wedge r) \Rightarrow (q \wedge r))$.

Solution:

$$
\begin{array}{ll}
 & (p \Rightarrow q) \Rightarrow ((p \wedge r) \Rightarrow (q \wedge r)) \\
= & \{ \text{ shunting (3.65) } \} \\
 & ((p \Rightarrow q) \wedge p \wedge r) \Rightarrow (q \wedge r) \\
= & \{ \text{ (3.66) } \} \\
 & (p \wedge q \wedge r) \Rightarrow (q \wedge r) \\
= & \{ \text{ weakening (3.76b) } \} \\
 & \mathit{True}
\end{array}
$$

19. Prove (4.3) $(p \Rightarrow q) \Rightarrow (p \wedge r \Rightarrow q \wedge r)$. Start with the consequent, since it has more structure.

Solution:

$$
\begin{array}{ll}
 & p \wedge r \Rightarrow q \wedge r \\
= & \{ \text{ definition of } \Rightarrow \text{ (3.59) } \} \\
 & \neg p \vee \neg r \vee (q \wedge r) \\
= & \{ \text{ distributivity (3.45) } \} \\
 & \neg p \vee ((\neg r \vee q) \wedge (\neg r \vee r)) \\
= & \{ \text{ excluded middle (3.28) and identity of } \wedge \text{ (3.39) } \} \\
 & \neg p \vee \neg r \vee q \\
\Leftarrow & \{ \text{ strengthening (3.76a) } \} \\
 & \neg p \vee q
\end{array}
$$

$$
\begin{array}{ll}
= & \{ \text{ definition of } \Rightarrow (3.59) \ \} \\
 & p \Rightarrow q
\end{array}
$$

20. Prove (3.76d) $p \vee (q \wedge r) \ \Rightarrow \ p \vee q$ using inequality reasoning. Start with the antecedent, since it has more structure, and use distributivity.

Solution:

$$
\begin{array}{ll}
 & p \vee (q \wedge r) \\
= & \{ \text{ distributivity } (3.45) \ \} \\
 & (p \vee q) \wedge (p \vee r) \\
\Rightarrow & \{ \text{ weakening } (3.76b) \ \} \\
 & p \vee q
\end{array}
$$

21. Prove $(p \Rightarrow q) \wedge (r \Rightarrow s) \ \Rightarrow \ (p \vee r \Rightarrow q \vee s)$ using inequality reasoning. **Hint**: first remove the implication in the antecedent, distribute as much as possible, and use (3.76d) and an absorption theorem.

Solution:

$$
\begin{array}{ll}
 & (p \Rightarrow q) \wedge (r \Rightarrow s) \\
= & \{ \text{ definition of } \Rightarrow (3.59) \ \} \\
 & (\neg p \vee q) \wedge (\neg r \vee s) \\
= & \{ \text{ distributivity } (3.46) \ \} \\
 & (\neg p \wedge \neg r) \vee (\neg p \wedge s) \vee (q \wedge \neg r) \vee (q \wedge s) \\
\Rightarrow & \{ \ (3.76d) \ \} \\
 & (\neg p \wedge \neg r) \vee s \vee (q \wedge \neg r) \vee q \\
= & \{ \text{ absorption } (3.43b) \ \} \\
 & (\neg p \wedge \neg r) \vee s \vee q \\
= & \{ \text{ de Morgan } (3.47b) \ \} \\
 & \neg(p \vee r) \vee s \vee q \\
= & \{ \text{ definition of } \Rightarrow (3.59) \ \} \\
 & p \vee r \Rightarrow q \vee s
\end{array}
$$

22. Prove (4.1) $p \Rightarrow (q \Rightarrow p)$ by assuming the antecedent.

Solution:
Assume: p

$$q \Rightarrow p$$

$$
\begin{aligned}
&= \quad \{ \text{ assumption } p \ \} \\
&\quad q \Rightarrow \mathit{True} \\
&= \quad \{ \text{ zero of } \Rightarrow (3.76b) \ \} \\
&\quad \mathit{True}
\end{aligned}
$$

23. Prove $(\neg p \Rightarrow q) \Rightarrow ((p \Rightarrow q) \Rightarrow q)$ by assuming the antecedent.

Solution:
Assume: $\neg p \Rightarrow q$

$$
\begin{aligned}
&\quad p \Rightarrow q \\
&= \quad \{ \text{ definition of } \Rightarrow (3.59) \ \} \\
&\quad \neg p \vee q \\
&\Rightarrow \quad \{ \text{ assumption, and monotonicity of } \vee \ (4.2) \ \} \\
&\quad q \vee q \\
&= \quad \{ \text{ idempotency of } \vee \ (3.26) \ \} \\
&\quad q
\end{aligned}
$$

24. Prove $(p \Rightarrow p') \wedge (q \Rightarrow q') \Rightarrow (p \vee q \Rightarrow p' \vee q')$ by assuming the antecedent.

Solution:
Assume: $(p \Rightarrow p')$ and $(q \Rightarrow q')$

$$
\begin{aligned}
&\quad p \vee q \\
&\Rightarrow \quad \{ \text{ assumption, and monotonicity of } \vee \ (4.2) \ \} \\
&\quad p' \vee q \\
&\Rightarrow \quad \{ \text{ assumption, and monotonicity of } \vee \ (4.2) \ \} \\
&\quad p' \vee q'
\end{aligned}
$$