Programming Languages: Imperative Program Construction
Practicals 3. Quantifications

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1. An integer array $X[0..N)$ is given, where $N \geqslant 1$. Explain, in words, what each of the following expressions mean.

   1. $b \equiv \langle \forall i : 0 \leqslant i < N : X[i] \geqslant 0 \rangle$.
   2. $r = \langle \# k : 0 \leqslant k < N : \langle \forall i : 0 \leqslant i < k : X[i] < X[k] \rangle \rangle$.
   3. $r = \langle \uparrow p, q : 0 \leqslant p \leqslant q \leqslant N \ \wedge\ \langle \forall i : p \leqslant i < q : X[i] > 0 \rangle : q - p \rangle$.
   4. $r = \langle \# p, q : 0 \leqslant p < q < N : X[p] = 0 \ \wedge\ X[q] = 1 \rangle$.
   5. $s = \langle \uparrow p, q : 0 \leqslant p < q < N : X[p] + X[q] \rangle$.
   6. $b \equiv \langle \forall p, q : 0 \leqslant p \ \wedge\ 0 \leqslant q \ \wedge\ p + q = N - 1 : X[p] = X[q] \rangle$.

Solution:

1. b equivals that all elements in X are non-negative (or, b is true if and only if all elements in X are non-negative).
2. r is the number of elements in X that are bigger than all elements preceding it.
3. r is the maximum length of positive segments in X.
4. r is the number of pairs of "a zero followed by a one" in X.
5. s is the maximum sum of any pairs of elements in X.
6. b equivals that X is a palindrome (that is, X reads the same backwards as forwards).

2. An integer array $X[0..N)$ is given, where $N \geqslant 1$. Express the following sentences in a formal way:

   1. r is the sum of the elements of X.
   2. X is increasing.
   3. all values of X are distinct.
   4. r is the length of a longest constant segment of X.
   5. r is the maximum of the sums of the segments of X.

Solution:

1. $r = \langle \Sigma i : 0 \leqslant i < N : X[i] \rangle$.

2. $\langle \forall i : 0 \leqslant i < N - 1 : X[i] < X[i+1] \rangle$.

3. $\langle \forall i, j : 0 \leqslant i < j < N : X[i] \neq X[j] \rangle$.

4. $r = \langle \uparrow p, q : 0 \leqslant p \leqslant q \leqslant N \wedge \langle \forall i, j : p \leqslant i < j < q : X[i] = X[j] \rangle : q - p \rangle$.

5. $r = \langle \uparrow p, q : 0 \leqslant p \leqslant q \leqslant N : \langle \Sigma i : p \leqslant i < q : X[i] \rangle \rangle$

3. Expand the following textual substitutions. If necessary, change the dummy, according to Dummy Renaming (8.21).

   1. $\langle \star x : 0 \leqslant x + r < n : x + v \rangle[v \backslash 3]$
   2. $\langle \star x : 0 \leqslant x + r < n : x + v \rangle[x \backslash 3]$
   3. $\langle \star x : 0 \leqslant x + r < n : x + v \rangle[n \backslash n + x]$
   4. $\langle \star x : 0 \leqslant x < r : \langle \star y : 0 \leqslant y : x + y + n \rangle \rangle[n \backslash x + y]$
   5. $\langle \star x : 0 \leqslant x < r : \langle \star y : 0 \leqslant y : x + y + n \rangle \rangle[r \backslash y]$

Solution:

1. $\langle \star x : 0 \leqslant x + r < n : x + 3 \rangle$
2. $\langle \star x : 0 \leqslant x + r < n : x + v \rangle$ (it is also okay to answer $\langle \star y : 0 \leqslant y + r < n : y + v \rangle$).
3. $\langle \star y : 0 \leqslant y + r < n + x : y + v \rangle$ (it is *not* okay to answer $\langle \star x : 0 \leqslant x + r < n + x : x + v \rangle$).
4. $\langle \star z : 0 \leqslant z < r : \langle \star w : 0 \leqslant w : z + w + x + y \rangle \rangle$
5. $\langle \star x : 0 \leqslant x < y : \langle \star y : 0 \leqslant y : x + y + n \rangle \rangle$ (renaming is not necessary in this case).

4. Prove the following theorems. Provided $0 \leqslant n$,

   (a) $\langle \Sigma i : 0 \leqslant i < n+1 : b[i] \rangle = b[0] + \langle \Sigma i : 1 \leqslant i < n+1 : b[i] \rangle$

Solution:

$$
\begin{array}{ll}
 & \langle \Sigma i : 0 \leqslant i < n+1 : b[i] \rangle \\
= & \{\ 0 \leqslant i < n+1 \ \equiv\ i = 0 \ \vee\ 1 \leqslant i < n+1\ \} \\
 & \langle \Sigma i : i = 0 \ \vee\ 1 \leqslant i < n+1 : b[i] \rangle \\
= & \{\ \text{range split (8.16), since } i = 0 \ \wedge\ 1 \leqslant i < n+1 \ \equiv\ \mathit{False}\ \} \\
 & \langle \Sigma i : i = 0 : b[i] = 0 \rangle + \langle \Sigma i : 1 \leqslant i < n+1 : b[i] \rangle \\
= & \{\ \text{one-point rule (8.14)}\ \} \\
 & b[0] + \langle \Sigma i : 1 \leqslant i < n+1 : b[i] \rangle
\end{array}
$$

   (b) $\langle \exists i : 0 \leqslant i < n+1 : b[i] = 0 \rangle = \langle \exists i : 0 \leqslant i < n : b[i] = 0 \rangle \vee b[n] = 0$

Solution:

$$\langle \exists i : 0 \leqslant i < n+1 : b[i] = 0 \rangle$$

$$
\begin{aligned}
&= \quad \{\ 0 \leqslant i < n+1 \ \equiv\ 0 \leqslant i < n \ \vee\ i = n \ \} \\
&\quad \langle \exists i : 0 \leqslant i < n \ \vee\ i = n : b[i] = 0 \rangle \\
&= \quad \{\ \text{range split (8.16), since } 0 \leqslant i < n \ \wedge\ i = n \ \equiv\ \mathit{False} \ \} \\
&\quad \langle \exists i : 0 \leqslant i < n : b[i] = 0 \rangle \vee \langle \exists i : i = n : b[i] = 0 \rangle \\
&= \quad \{\ \text{one-point rule (8.14)} \ \} \\
&\quad \langle \exists i : 0 \leqslant i < n : P \rangle \vee b[n] = 0
\end{aligned}
$$

5. Prove that $\langle \forall x : R : P \rangle \equiv P \vee \langle \forall x :: \neg R \rangle$, provided $\neg \mathit{occurs}(x, P)$.

Solution:

$$
\begin{aligned}
&\quad P \vee \langle \forall x :: \neg R \rangle \\
&= \quad \{\ \text{distributivity, since } \neg \mathit{occurs}(x, P) \ \} \\
&\quad \langle \forall x :: P \vee \neg R \rangle \\
&= \quad \{\ P \vee \neg R \ \equiv\ R \Rightarrow P, \text{ trading} \ \} \\
&\quad \langle \forall x : R : P \rangle
\end{aligned}
$$

6. Prove the *range weakening* rule: $\langle \forall x : Q \vee R : P \rangle \Rightarrow \langle \forall x : Q : P \rangle$.

Solution:

$$
\begin{aligned}
&\quad \langle \forall x : Q \vee R : P \rangle \\
&= \quad \{\ \text{range split (8.18), since } \wedge \text{ idempotent} \ \} \\
&\quad \langle \forall x : Q : P \rangle \wedge \langle \forall x : R : P \rangle \\
&\Rightarrow \quad \{\ \text{weakening (3.76b)} \ \} \\
&\quad \langle \forall x : Q : P \rangle
\end{aligned}
$$

7. Prove the *body weakening* rule: $\langle \forall x : R : P \wedge Q \rangle \Rightarrow \langle \forall x : R : P \rangle$.

Solution:

$$
\begin{aligned}
&\quad \langle \forall x : R : P \wedge Q \rangle \\
&= \quad \{\ \text{distributivity, since } P, Q : \mathit{Bool} \ \} \\
&\quad \langle \forall x : R : P \rangle \wedge \langle \forall x : R : Q \rangle \\
&\Rightarrow \quad \{\ \text{weakening (3.76b)} \ \} \\
&\quad \langle \forall x : R : P \rangle
\end{aligned}
$$