Axioms and Theorems of the Propositional Calculus

In *A Logical Approach to Discrete Math*, D. Gries and F. B. Schneider. Springer, 1994.

(1.1) **Substitution**: $\dfrac{E}{E[v \backslash F]}$

(1.4) **Transitivity**: $\dfrac{X = Y \qquad Y = Z}{X = Z}$

(1.5) **Leibniz**: $\dfrac{X = Y}{E[z \backslash X] = E[z \backslash Y]}$

3.1 Equivalence and True

$*(3.1)$ **Axiom, Associativity of** $\equiv$:
$$((p \equiv q) \equiv r) \;\equiv\; (p \equiv (q \equiv r))$$

$*(3.2)$ **Axiom, Symmetry of** $\equiv$: $p \equiv q \equiv q \equiv p$

$*(3.3)$ **Axiom, Identity of** $\equiv$: $\mathit{True} \equiv p \equiv p$

$*(3.4)$ True

$*(3.5)$ **Reflexivity of** $\equiv$**:** $p \equiv p$

3.2 Negation, Inequivalence, and False

$*(3.15)$ **Axiom, Definition of** False : $\neg p \equiv p \equiv \mathit{False}$

(3.10) **Axiom, Definition of** $\not\equiv$: $(p \not\equiv q) \equiv \neg(p \equiv q)$

Theorems Relating $\equiv$, $\not\equiv$, $\neg$

(3.8) $\mathit{False} \equiv \neg \mathit{True}$

$*(3.9)$ **Distributivity of** $\neg$ **over** $\equiv$:
$$\neg(p \equiv q) \equiv \neg p \equiv q$$

(3.11) $\neg p \equiv q \equiv p \equiv \neg q$

$*(3.12)$ **Double negation** : $\neg\neg p \equiv p$

(3.13) **Negation of** False : $\neg \mathit{False} \equiv \mathit{True}$

(3.14) $(p \not\equiv q) \equiv \neg p \equiv q$

$*(3.16)$ **Symmetry of** $\not\equiv$: $(p \not\equiv q) \equiv (q \not\equiv p)$

$*(3.17)$ **Associativity of** $\not\equiv$:
$$((p \not\equiv q) \not\equiv r) \equiv (p \not\equiv (q \not\equiv r))$$

(3.18) **Mutual associativity** :
$$((p \not\equiv q) \equiv r) \equiv (p \not\equiv (q \equiv r))$$

(3.19) **Mutual interchangeability** :
$$p \not\equiv q \equiv r \;\equiv\; p \equiv q \not\equiv r$$

3.3 Disjunction

$*(3.24)$ **Axiom, Symmetry of** $\vee$: $p \vee q \equiv q \vee p$

$*(3.25)$ **Axiom, Associativity of** $\vee$:
$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

$*(3.26)$ **Axiom, Idempotency of** $\vee$: $p \vee p \equiv p$

$*(3.27)$ **Axiom, Distributivity of** $\vee$ **over** $\equiv$:
$$p \vee (q \equiv r) \equiv p \vee q \equiv p \vee r$$

$*(3.28)$ **Axiom, Excluded Middle** : $p \vee \neg p$

Basic Properties of $\vee$

$*(3.29)$ **Zero of $\vee$:** $p \vee \mathit{True} \equiv \mathit{True}$
$*(3.30)$ **Identity of $\vee$:** $p \vee \mathit{False} \equiv p$
$*(3.31)$ **Distributivity of $\vee$ over $\vee$:**
$$p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$$
(3.32) $p \vee q \equiv p \vee \neg q \equiv p$

3.4 Conjunction

(3.35) **Axiom, Golden rule :** $p \wedge q \equiv p \equiv q \equiv p \vee q$

Basic Properties of $\wedge$

$*(3.36)$ **Symmetry of $\wedge$:** $p \wedge q \equiv q \wedge p$
$*(3.37)$ **Associativity of $\wedge$:**
$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$
$*(3.38)$ **Idempotency of $\wedge$:** $p \wedge p \equiv p$
$*(3.39)$ **Identity of $\wedge$:** $p \wedge \mathit{True} \equiv p$
$*(3.40)$ **Zero of $\wedge$:** $p \wedge \mathit{False} \equiv \mathit{False}$
(3.41) **Distributivity of $\wedge$ over $\wedge$:**
$$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge (p \wedge r)$$
$*(3.42)$ **Contradiction :** $p \wedge \neg p \equiv \mathit{False}$

Theorems relating $\wedge$ and $\vee$

$*(3.43)$ **Absorption :** (a) $p \wedge (p \vee q) \equiv p$
(b) $p \vee (p \wedge q) \equiv p$
(3.44) **Absorption :** (a) $p \wedge (\neg p \vee q) \equiv p \wedge q$
(b) $p \vee (\neg p \wedge q) \equiv p \vee q$
$*(3.45)$ **Distributivity of $\vee$ over $\wedge$:**
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$
$*(3.46)$ **Distributivity of $\wedge$ over $\vee$:**
$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$
$*(3.47)$ **De Morgan :** (a) $\neg(p \wedge q) \equiv \neg p \vee \neg q$
(b) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Theorems relating $\wedge$ and $\equiv$

(3.48) $p \wedge q \equiv p \wedge \neg q \equiv \neg p$
(3.49) $p \wedge (q \equiv r) \equiv p \wedge q \equiv p \wedge r \equiv p$
(3.50) $p \wedge (q \equiv p) \equiv p \wedge q$
(3.51) **Replacement :**
$$(p \equiv q) \wedge (r \equiv p) \;\equiv\; (p \equiv q) \wedge (r \equiv q)$$

Alternative Definitions of $\equiv$ and $\not\equiv$

(3.52) **Definition of $\equiv$:** $p \equiv q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
(3.53) **Exclusive or :** $p \not\equiv q \equiv (\neg p \wedge q) \vee (p \wedge \neg q)$

3.5 Implication

$*(3.57)$ **Axiom, Definition of implication :**
$$p \Rightarrow q \;\equiv\; p \vee q \equiv q$$
(3.58) **Axiom, Consequences :** $p \Leftarrow q \equiv q \Rightarrow p$

Rewriting Implication

$*(3.59)$ **Definition of implication :**
$$p \Rightarrow q \;\equiv\; \neg p \vee q$$
$*(3.60)$ **Definition of implication :**
$$p \Rightarrow q \;\equiv\; p \equiv p \wedge q$$
(3.61) **Contrapositive :** $p \Rightarrow q \;\equiv\; \neg q \Rightarrow \neg p$

Miscellaneous Theorems About Implication

(3.62) $p \Rightarrow (q \equiv r) \equiv p \wedge q \equiv p \wedge r$
(3.63) **Distributivity of $\Rightarrow$ over $\equiv$:**
$$p \Rightarrow (q \equiv r) \equiv p \Rightarrow q \equiv p \Rightarrow r$$
(3.64) $p \Rightarrow (q \Rightarrow r) \;\equiv\; (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$
(3.65) **Shunting :** $p \wedge q \Rightarrow r \;\equiv\; p \Rightarrow (q \Rightarrow r)$
(3.66) $p \wedge (p \Rightarrow q) \equiv p \wedge q$
(3.67) $p \wedge (q \Rightarrow p) \equiv p$
(3.68) $p \vee (p \Rightarrow q) \equiv \mathit{True}$
(3.69) $p \vee (q \Rightarrow p) \equiv q \Rightarrow p$
(3.70) $p \vee q \Rightarrow p \wedge q \;\equiv\; p \;\equiv\; q$

Implication and Boolean Constants

$*(3.71)$ **Reflexivity of** $\Rightarrow$: $p \Rightarrow p \equiv True$

$*(3.72)$ **Right zero of** $\Rightarrow$: $p \Rightarrow True \equiv True$

$*(3.73)$ **Left identity of** $\Rightarrow$: $True \Rightarrow p \equiv p$

$*(3.74)$ $p \Rightarrow False \equiv \neg p$

$*(3.75)$ $False \Rightarrow p \equiv True$

Weakening, Strengthening, and Modus Ponens

(3.76) **Weakening, Strengthening** :

$*(a)$ $p \Rightarrow p \vee q$

$*(b)$ $p \wedge q \Rightarrow p$

(c) $p \wedge q \Rightarrow p \vee q$

(d) $p \vee (q \wedge r) \Rightarrow p \vee q$

(e) $p \wedge q \Rightarrow p \wedge (q \vee r)$

(3.77) **Modus ponens** : $p \wedge (p \Rightarrow q) \Rightarrow q$

Forms of Case Analysis

(3.78) $(p \Rightarrow r) \wedge (q \Rightarrow r) \equiv (p \vee q \Rightarrow r)$

(3.79) $(p \Rightarrow r) \wedge (\neg p \Rightarrow r) \equiv r$

Mutual Implication and Transitivity

(3.80) **Mutual implication** :

$$(p \Rightarrow q) \wedge (q \Rightarrow p) \;\equiv\; p \equiv q$$

(3.81) **Antisymmetry** :

$$(p \Rightarrow q) \wedge (q \Rightarrow p) \Rightarrow (p \equiv q)$$

(3.82) **Transitivity** :

(a) $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$

(b) $(p \equiv q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$

(c) $(p \Rightarrow q) \wedge (q \equiv r) \Rightarrow (p \Rightarrow r)$

Leibniz's Rule as an Axiom

(3.83) **Axiom, Leibniz** :

$$(e = f) \Rightarrow (E[z\backslash e] = E[z\backslash f])$$

Rules of Substitution

(3.84) **Substitution** :

(a) $(e = f) \wedge E^z_e \;\equiv\; (e = f) \wedge E^z_f$

(b) $(e = f) \Rightarrow E^z_e \;\equiv\; (e = f) \Rightarrow E^z_f$

(c) $q \wedge (e = f) \Rightarrow E^z_e \;\equiv\; q \wedge (e = f) \Rightarrow E^z_f$

Replacing Variables by Boolean Constants

(3.85) **Replace by** $True$:

(a) $p \Rightarrow E^z_p \;\equiv\; p \Rightarrow E^z_{True}$

(b) $q \wedge p \Rightarrow E^z_p \;\equiv\; q \wedge p \Rightarrow E^z_{True}$

(3.86) **Replace by** $False$:

(a) $E^z_p \Rightarrow p \;\equiv\; E^z_{False} \Rightarrow p$

(b) $E^z_p \Rightarrow p \vee q \;\equiv\; E^z_{False} \Rightarrow p \vee q$

(3.87) **Replace by** $True$: $p \wedge E^z_p \;\equiv\; p \wedge E^z_{True}$

(3.88) **Replace by** $False$: $p \vee E^z_p \;\equiv\; p \vee E^z_{False}$

(3.89) **Shannon** :

$$E^z_p \;\equiv\; (p \wedge E^z_{True}) \vee (\neg p \wedge E^z_{False})$$

4.1 An Abbreviation for Proving Implications

(4.1) $p \Rightarrow (q \Rightarrow p)$

(4.2) **Monotonicity of** $\vee$:

$$(p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r)$$

(4.3) **Monotonicity of** $\wedge$:

$$(p \Rightarrow q) \Rightarrow (p \wedge r \Rightarrow q \wedge r)$$

4.2 Additional Proof Techniques

(4.4) **(Extended) Deduction Theorem**. Suppose adding $P_1, \ldots P_n$ as axioms (with the variables of each P_i considered to be constants) allows Q to be proved. Then $P_1 \wedge \ldots P_n \Rightarrow Q$ is a theorem.